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Mathematical and Scientific Analysis of European and Chinese Tuning Systems

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ABSTRACT

This paper presents a comparative mathematical and scientific analysis of European and Chinese tuning systems, tracing how cultures approached the problem of dividing the octave and balancing consonance with flexibility. Using frequency ratios and logarithmic pitch units (cents), it quantifies interval sizes in Pythagorean tuning, just intonation (5-limit), meantone temperaments, and modern 12-tone equal temperament (12-TET), with worked examples and tables to show the trade-offs among pure intervals, "wolf" intervals, and modulatory freedom. The study then examines traditional Chinese theory of the 十二律 *(twelve lü) and the* 三分損益 *(cycle of fifths) method, its pentatonic emphasis, and Jing Fang's near-equivalence of 53 fifths to 31 octaves—revealing deep historical parallels with Western developments. Acoustical foundations (harmonic overtones, resonance, beating) explain why small-integer ratios sound consonant and how tempering subtly detunes them to enable practical performance across keys. The analysis concludes that, despite differing musical priorities—European polyphony versus Chinese pentatonic practice—both traditions ultimately converged on equal temperament as a universal compromise between purity and versatility, highlighting the shared interplay of mathematics, physics, and musical aesthetics.*

KEYWORDS: *Tuning systems, Pythagorean, just intonation, meantone, equal temperament, twelve lü, sanfen sunyi, cents, consonance, resonance.*

INTRODUCTION

Musical tuning systems determine the exact frequencies of notes and intervals. Different cultures developed distinct tuning approaches based on mathematical ratios and acoustic principles. This paper examines European tuning systems – from ancient Pythagorean tuning through just intonation, meantone temperaments, to modern 12-tone equal temperament – in comparison with traditional Chinese tuning methods like *sanfen sunyi* (the "three-part subtracting and adding" cycle of fifths) and the pentatonic scale. We explore the mathematical foundations (frequency ratios, logarithmic pitch scales, interval calculations) and scientific principles (acoustics, resonance, harmonic overtones) underlying these systems and provide tables and worked examples to quantify interval sizes and trade-offs. The goal is a deep analytical view of how tuning is structured and understood in both European and Chinese musical science, supported by historical and theoretical evidence.

MATHEMATICAL FOUNDATIONS OF TUNING

Frequency Ratios and Intervals: At the core of tuning theory is the idea that musical intervals correspond to frequency ratios of small integers. The simplest example is the octave, a doubling of frequency with ratio 2:1. The next most fundamental interval is the perfect fifth with ratio 3:2[1]. Pythagorean tradition (both European and Chinese) chose the 3:2 fifth as the "generator" for scales because it is the next simplest ratio after the octave, corresponding to the third harmonic of a string (hence a very consonant interval) [2]. In any tuning system based on pure intervals, combining intervals means multiplying their frequency ratios. For example, a perfect fifth (3:2) above a middle C (say 256 Hz) gives G at 256 × 3/2 = 384 Hz; a perfect fourth (4:3) above that G returns to the octave C: 384 × 4/3 = 512 Hz, exactly double 256 Hz. This demonstrates how 3:2 and 4:3 (a fifth up and a fourth up) complement each other to span an octave (3/2 × 4/3 = 2) in a pure ratio system[3].

Logarithmic Pitch Scaling (Cents):

Human perception of pitch is approximately logarithmic, making it mathematically convenient to measure intervals on a logarithmic scale. Modern tunings use the cent unit: 1 octave = 1200 cents by definition, so one equal-tempered semitone is 100¢[3][4]. The size in cents of any interval with frequency ratio r is given by:

$$C = 1200 \times \log_2(r)$$

For example, a perfect fifth (r = 3/2) is:

$C = 1200 \times \log_2(3/2) \approx 701.96$ cents, which is very close to 702 cents.

In twelve-tone equal temperament (12-TET), the octave is divided into 12 exactly equal semitones of 100 cents each. The frequency of the nth semitone above a reference pitch f_0 is given by:

$$f_n = f_0 \times 2^{n/12}$$

Here, $2^{1/12} \approx 1.059463$ is the constant ratio between adjacent semitones[4]. This exponential spacing means pitch relationships become additive in cents (or semitone units), which greatly simplifies modulation (changing keys).

The table 1 below compares three tuning systems—Equal Temperament, Pythagorean Tuning, and Just Intonation—for the C major scale, showing both frequencies (Hz) and pitch positions in cents relative to C.

Table 1. Equal Temperament, Pythagorean Tuning, and Just Intonation

Note	**Equal Temperament (Hz)**	**Equal Temperament (¢)**	**Pythagorean (Hz)**	**Pythagorean (¢)**	**Just Intonation (Hz)**	**Just Intonation (¢)**
C4	261.626	0	261.626	0.0	261.626	0.0
D4	293.665	200	294.329	203.91	294.329	203.91
E4	329.628	400	331.12	407.82	327.032	386.31
F4	349.228	500	348.834	498.04	348.834	498.04
G4	391.995	700	392.438	701.96	392.438	701.96
A4	440.0	900	441.493	905.87	436.043	884.36
B4	493.883	1100	496.68	1109.78	490.548	1088.27
C5	523.251	1200	523.251	1200.0	523.251	1200.0

Worked Examples

- Perfect fifth (pure): $r = 3/2 \rightarrow C = 1200 \times \log_2(3/2) \approx 701.96$¢.
- Perfect fifth (12-TET): $r = 2^{(7/12)} \approx 1.4983 \rightarrow 700.0$¢.
- Major third (pure): $r = 5/4 = 1.25 \rightarrow \approx 386.3$¢.
- Major third (12-TET): $r = 2^{(4/12)} = 2^{1/3} \approx 1.2599 \rightarrow 400.0$¢.

The equal-tempered perfect fifth is 700.0¢ (ratio $2^{(7/12)} \approx 1.4983$), slightly flatter than the pure 3:2 ≈ 701.96¢; the equal-tempered major third is 400.0¢ (ratio $2^{(4/12)} = 2^{1/3} \approx 1.2599$), slightly sharper than pure 5:4 ≈ 386.3¢.

INTERVAL COMPUTATION

In ratio-based tuning, intervals are calculated by multiplying or dividing known frequency ratios. For example, a major third in just intonation has a ratio of 5:4, meaning its frequency is 1.25 times that of the root note[5]. Similarly, a minor third has a ratio of 6:5, or 1.2 times the root frequency. To combine intervals, the ratios are multiplied. For instance, multiplying a whole tone (9:8) by another whole tone (9:8) gives:

$$(9/8) \times (9/8) = 81/64 \approx 1.2656$$

This result corresponds to a Pythagorean major third, which is approximately 407.8 cents.

In logarithmic terms, interval sizes can be added or subtracted directly in cents. For example:

Perfect fifth (≈702¢) – Perfect fourth (≈498¢) = Major second (≈204¢)

Using cents in this way allows for straightforward comparison of interval sizes across different tuning systems. This approach will be applied in our analysis to compare European and Chinese tunings.

ACOUSTIC PRINCIPLES AND HARMONIC RESONANCE

Consonant intervals arise from the harmonic overtone series: a vibrating string or air column produces partials at integer multiples of a fundamental frequency f_0[6].

Example: If the fundamental is C_2 at $f_0 = 100$ Hz, then:

1st harmonic: $f_1 = 1 \times f_0 = 100$ Hz (C_2)

2nd harmonic: $f_2 = 2 \times f_0 = 200$ Hz (C_3)

3rd harmonic: $f_3 = 3 \times f_0 = 300$ Hz (G_3)

4th harmonic: $f_4 = 4 \times f_0 = 400$ Hz (C_4)

5th harmonic: $f_5 \approx 5 \times f_0 = 500$ Hz (E_4)[7]

These naturally occurring harmonics correspond to simple frequency ratios:

Octave: 2:1

Perfect fifth: 3:2

Perfect fourth: 4:3

Major third: 5:4

Small-integer ratios sound smooth because two waveforms realign periodically. In a perfect fifth 3:2 (e.g., 200 Hz vs. 300 Hz), the periods are $T_1 = 1/200 = 0.005$ s and $T_2 = 1/300 \approx 0.00333$ s; they coincide every 0.01 s ($\mathrm{LCM}(T_1, T_2) = 0.01$s). The tones also share harmonics—here, 600 Hz (the 3rd harmonic of 200 Hz and the 2nd of 300 Hz)—which reinforces the blend and minimizes beating [8]. By contrast, intervals with complex or inharmonic ratios do not align periodically; their waveforms never fully sync up, leading to audible beats or roughness.

Modern acoustics confirms that simple frequency ratios underpin consonance[9]. Notes related by ratios like 2:1, 3:2, 4:3, 5:4 align with overlapping overtones, creating stability. Psychoacoustic phenomena, such as combination tones, enhance this effect—for example, a perfect fifth (C–G) may produce a difference tone an octave below the lower note, reinforcing the harmony[14]. In contrast, Dissonant spans such as the tritone generate combination tones that fail to integrate harmonically.

Historically, tuning systems have leveraged these facts. Just intonation and Pythagorean tuning preserve selected pure ratios to maximize consonance in key intervals, whereas tempered systems deliberately nudge those ratios to trade a bit of purity for the ability to modulate freely among keys.

RESONANCE, INSTRUMENT PHYSICS, AND EUROPEAN TUNING SYSTEMS

Musical instruments are designed around natural resonances. A string or air column tuned to frequency f will resonate at f and its harmonics $2f$, $3f$, $4f$, etc. Thus, when two notes are in a pure ratio, one can excite resonances in another instrument tuned to a harmonic of it. For example, if a piano's A string (220 Hz) is struck, a properly tuned E (a fifth above, ~330 Hz) on another instrument might resonate subtly, because 330 Hz is the 3rd harmonic of 110 Hz (half of 220 Hz) and aligns with the overtone series of A.

A piano string A have: $f_A = 220$ Hz

A note E a perfect fifth above has: $f_E = (3/2) \times 220 = 330$ Hz

On another instrument, 330 Hz is the 3rd harmonic of: $f_{base} = (1/2) \times 220 = 110$ Hz

In equal temperament, the perfect fifth is slightly narrower than pure: $r_5(ET) = 2^{(7/12)} \approx 1.4983$ (700.0¢) versus the pure 3/2 = 1.5 (≈701.96¢). This small misalignment weakens sympathetic resonance slightly and introduces gentle beating—one reason ensembles with flexible pitch adjust by ear. Likewise, string quartets often lower their major thirds from the ET ratio $2^{(4/12)} = 2^{1/3} \approx 1.2599$ (400.0¢) toward the pure 5/4 = 1.25 (≈386.3¢) in cadences to maximize consonance.

In summary, the scientific ideal in tuning is to match intervals to harmonic-series ratios to maximize consonance and resonance. Constraints of musical practice, however, sometimes require detuning those pure ratios – and the contrast between European and Chinese tuning histories largely centers on how each tradition managed this consonance-versus-flexibility tradeoff.

European Tuning Systems: From Pure to Tempered

European music theory has evolved through several tuning systems, each balancing pure harmonies with the ability to play in multiple keys. Pythagorean Tuning (3-limit tuning) uses only the prime factors 2 and 3, stacking pure fifths (3:2) and adjusting octaves (2:1) to bring pitches into a single octave range[1][2]. For example, starting from C:

G = C × (3/2)

D = G × (3/2) = C × (9/8)

A = D × (3/2) = C × (27/16) ...

Continuing this process yields a 7-note diatonic scale built from pure fifths.

Pythagorean tuning produces perfectly pure fifths and fourths:

Perfect fifth: 3/2 = 1.5 → 702.0¢

Perfect fourth: 4/3 ≈ 1.3333 → 498.0¢

Whole tone: 9/8 = 1.125 → 203.9¢

However, its major thirds and sixths are sharper than just intonation values:

Pythagorean major third: 81/64 ≈ 1.265625 → 407.8¢ (vs. Just major third: 5/4 = 1.25 → 386.3¢) and the Difference: 407.8¢ – 386.3¢ ≈ +21.5¢

Pythagorean major sixth: 27/16 ≈ 1.6875 → 905.9¢ (vs. Just major sixth: 5/3 ≈ 1.6667 → 884.4¢) and the Difference: 905.9¢ – 884.4¢ ≈ +21.5¢

The sharpening comes from stacking fifths to form a third instead of using the harmonic 5th partial. In medieval practice, such thirds were treated as dissonances.

The Pythagorean comma arises because twelve fifths overshoot seven octaves: $(3/2)^{12} \approx 129.7463$, while $2^7 = 128$ → ratio difference = $(3/2)^{12} \div 2^7 \approx 1.01364$ → 23.46¢. This forces one interval, called the 'wolf fifth', to be narrowed significantly—often to about 680¢—to close the circle of fifths. The resulting 12 semitones are unequal: there are seven smaller diatonic semitones (~90.22¢) and five larger chromatic semitones (~113.68¢)[1][3][4].

Compared with 12TET:

Minor second: Pythagorean ≈ 90.22¢ vs. 12-TET = 100¢

Major second: Pythagorean ≈ 203.91¢ vs. 12-TET = 200¢

Major third: Pythagorean ≈ 407.82¢ vs. 12-TET = 400¢

Perfect fifth: Pythagorean ≈ 701.96¢ vs. 12-TET = 700¢

Tritone discrepancy:

Augmented fourth: 729/512 ≈ 1.42578 → 611.73¢

Diminished fifth: 1024/729 ≈ 1.40466 → 588.27¢

Difference: 611.73¢ – 588.27¢ ≈ 23.46¢ (Pythagorean comma)

In 12-TET, the tritone is fixed at exactly 600¢, eliminating the split.

Figure 1 illustrates the difference: equal temperament has uniformly 100¢ semitones, whereas Pythagorean tuning alternates smaller and larger step sizes. Despite these issues, Pythagorean tuning was satisfactory for medieval monophonic and modal music. Pure fifths gave melodies a resonant hollowness, and since harmony (chords) was not yet complex, the wolf interval could be avoided by staying in certain keys. Melodically, the unequal step sizes produced by 3:2 cycles lent medieval modes a distinctive character. When polyphonic harmony expanded in the Renaissance to treat thirds as consonances, the sharp Pythagorean third became problematic[5], leading to new tuning strategies.

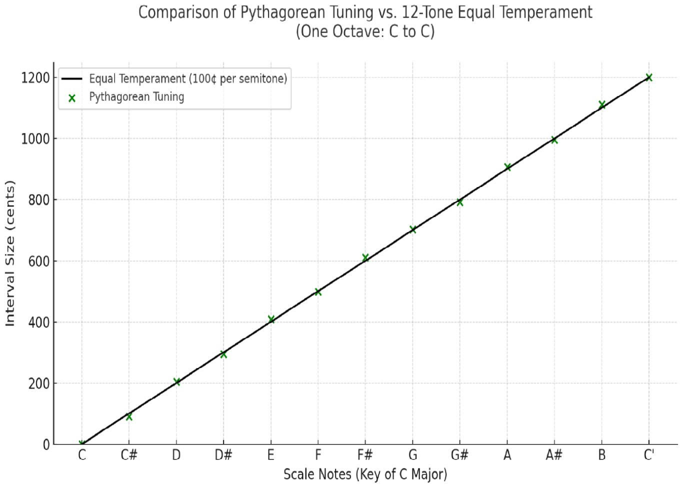

Figure 1. Comparison of Pythagorean tuning vs. 12-tone equal-tempered intervals (one octave, C to C). The horizontal axis shows the scale notes in the key of C major; the vertical axis shows interval size in cents (1200¢ = one octave). The black line indicates the equal-tempered scale (each semitone = 100¢). Green points show the Pythagorean tuning intervals based on pure fifths.

Compared with 12-tone equal temperament (12-TET), which fixes every semitone at exactly 100¢, Pythagorean tuning has more variation. Minor seconds (~90¢) are noticeably smaller, major seconds (~204¢) slightly larger, and the major third (~408¢) is sharper than the equal-tempered 400¢. The perfect fifth (~702¢) remains nearly pure, only about 2¢ sharper than in 12-TET. The largest difference occurs at the tritone. In 12-TET, the tritone is a single 600¢ interval. In Pythagorean tuning, it splits into an augmented fourth (≈611.7¢) and a diminished fifth (≈588.3¢), differing by the Pythagorean comma (~23.5¢)[11]. Equal temperament merges F♯ and Gb into one pitch, eliminating the comma difference. Other intervals differ by only about 2–10¢, small enough to be acceptable in most performance contexts. This explains why 12-TET became a successful compromise: it tempers all intervals slightly so that music can be played in all keys without intolerable dissonance.

JUST INTONATION (5-LIMIT TUNING)

Just intonation incorporates not only the pure perfect fifth (3:2) but also the pure major third (5:4), along with related intervals such as the minor third (6:5) and major sixth (5:3), all derived from the harmonic series[7][8]. In a just diatonic scale—described by Gioseffo Zarlino in 1558, though its origins go back to Ptolemy—the major triads are tuned perfectly. For example, in a C major chord (C–E–G):

C:E = 5:4 → major third = 386.3¢ (pure)

C:G = 3:2 → perfect fifth = 702.0¢ (pure)

These ratios yield extremely smooth consonances. The just major third at 386.3¢ produces far less beating than the Pythagorean major third at 407.8¢.

However, the system cannot keep all fifths at the pure 3:2 ratio if the major thirds are set at 5:4. In one just-tuned C scale, for example:

A = 5:3 (major sixth above C)

D = 9:8 (major second above C)

Fifth D:A = 40:27 → ≈ 680¢ (narrow, poor consonance)

This problem arises because introducing pure thirds creates additional commas, most notably the syntonic comma:

Syntonic comma = 81:64 vs. 5:4 → ≈ 21.51¢

To preserve just major thirds, some fifths must be narrowed by this comma, making them less consonant.

In practice, just intonation works beautifully within a single key but becomes unstable if the music changes key or uses chords outside the home tonality. Each key requires its own adjustments—either extra pitches for enharmonic equivalents or physical retuning—which is impractical for fixed-pitch instruments across many keys.

MEANTONE TEMPERAMENT

During the Renaissance, meantone temperaments, especially quarter-comma meantone, became widely used. The term ‘meantone’ refers to tuning each whole tone so that it lies midway between pure intervals. In quarter-comma meantone, each perfect fifth is flattened by one quarter of the syntonic comma—about (1/4 × 21.51¢ ≈ 5.38¢)—resulting in a fifth measuring roughly 696.6¢ instead of the pure 702¢. This adjustment produces a major third of exactly 386.3¢, which corresponds precisely to the pure 5:4 ratio, because it is formed from two tempered whole tones of approximately 193.16¢ each[12].

This tuning created exceptionally consonant triads, perfectly suited to the smoother harmonic language of the 16th and 17th centuries. However, the trade-off was that traveling far around the circle of fifths introduced an even more severe ‘wolf’ interval than in Pythagorean tuning. Depending on how the temperament was arranged, the wolf fifth could be as large as 737¢ or as small as 648¢, making remote keys practically unusable[10]. Furthermore, enharmonic notes such as G♯ and Ab were tuned differently, so they were not interchangeable. For example, G♯ tuned as the major third above E could differ noticeably from Ab derived from a chain of fifths starting on C. To address this, some keyboard instruments were built with split keys to provide separate pitches for such enharmonic equivalents.

Meantone temperament represented an effective compromise in the late Renaissance and early Baroque periods for music that did not modulate far from its home key. It maintained the purity of the most frequently used intervals — fifths and thirds — while sacrificing the usability of distant keys. This balance between consonance in common harmonies and dissonance in remote ones reflects the musical priorities of its time.

EQUAL TEMPERAMENT (12-FET)

By the late 18th century, well temperaments allowed performance in all keys (with differing key colors). Through the 19th–20th centuries, 12tone equal temperament (12TET) became the standard that makes all keys equivalent[11]. In 12TET each semitone is 100¢, and only the octave is pure. The perfect fifth is $2^{(7/12)} \approx 1.4983 \rightarrow 700.0$¢ (≈1.96¢ flatter than the pure 3:2 ≈ 701.96¢); the major third is $2^{(4/12)} = 2^{1/3} \approx 1.2599 \rightarrow 400.0$¢ (≈13.69¢ sharper than pure 5:4 ≈ 386.3¢). These small, even offsets remove wolf intervals and enable uniform modulation. Historically, the mathematics of equal division of the octave was worked out independently by Zhu Zaiyu (1584) and Simon Stevin (1585); universal adoption followed much later with modern instrument making and ensemble practice[12].

In equal temperament, intervals are defined by powers of 2, divided into 12 equal semitones.

Perfect fifth(12-TET): $2^{7/12} \approx 1.4983 \rightarrow 700.0$¢ (pure = 3/2 → 701.96¢)

Major third(12-TET): $2^{4/12} = 2^{1/3} \approx 1.2599 \rightarrow 400$¢ (pure = 5/4 → 386.31¢)

These values are irrational numbers, meaning the intervals no longer correspond to small whole-number ratios — a sharp break from Pythagorean and just intonation systems[14].

Because 12TET intervals are powers of 2 split into 12 equal steps, their ratios are irrational and no longer match simple small integer relationships. As a result, partials do not align perfectly with the harmonic series. This produces a mild beating in chords that is absent in pure just intonation[13]. Over time, listeners have adapted so that equal-tempered intervals still sound consonant, though with a subtly different tone color. Crucially, equal temperament was the first systematic tuning to allow free modulation and the use of all 24 major and minor keys on a single instrument without retuning. This consistency led it to become the dominant Western tuning system since the 18th century[11].

Table 2. Comparison of Equal Temperament vs. Pure Tuning

Interval	Equal Temperament (Ratio)	Pure Tuning (Ratio)	Difference
Perfect Fifth	$2^{7/12} \approx 1.4983$	3/2 = 1.5	≈ -0.0017
Major Third	$2^{1/3} \approx 1.2599$	5/4 = 1.25	≈ +0.0099

SUMMARY OF EUROPEAN SYSTEMS

Early European tuning began with Pythagorean practice, which prioritizes pure 3:2 fifths (≈ 701.96¢) but yields wide major thirds at 81:64 (≈ 407.8¢); the accumulation of pure fifths introduces the Pythagorean comma (≈ 23.46¢), producing wolf intervals in certain keys. As harmonic writing expanded, 5-limit just intonation sought purer sonorities by using 5:4 major thirds (≈ 386.31¢) alongside 3:2 fifths, but the syntonic comma (81:80 ≈ 21.51¢) made fixed-key usage and modulation difficult. Meantone temperaments then tempered the fifths to favor pure thirds—most famously, quarter-comma meantone narrows each fifth by ≈ 5.38¢ to about 696.6¢—which improves local consonance while degrading remote keys. Ultimately, 12-tone equal temperament divides the octave into equal semitones ($2^{1/12} \approx 1.059463$, set to 100¢ exactly); in this system the fifth is 700.0¢ and the major third is 400.0¢, eliminating wolf intervals, aligning enharmonic spellings, and creating key equivalence around the circle of fifths. The arc from small-integer ratios (3/2, 5/4) to the irrational roots of two reflects a conscious trade-off—recognized by the nineteenth century—exchanging a measure of pure consonance for the versatility required by frequent modulation and complex harmonic progressions[1].

TRADITIONAL CHINESE TUNING (十二律) & SANFEN SUNYI

Chinese music theory developed its own framework for pitch, rooted in nature and philosophy, yet mathematically similar to the Western Pythagorean approach. The ancient system centers on the 十二律 (shí'èr lǜ) or 12 lü pitch pipes – twelve fundamental pitches analogous to the chromatic scale. According to tradition, these were generated by the scholar Ling Lun (c. 3rd millennium BCE, mythology) and later formalized in texts like the Lüshi Chunqiu (c. 239 BCE).

The method for generating the 12 lü is called 三分損益 (sānfēn sǔnyì), meaning "three-part subtracting and adding," which is mathematically equivalent to the cycle of fifths. Starting from the fundamental pitch (the first lü, Huángzhōng 黄鐘 or 'Yellow Bell'):

- Subtract one third of the tube length → length = (2/3) × original → frequency × (3/2) → perfect fifth up. • Add one third of the tube length → length = (4/3) × original → frequency × (3/4) → perfect fourth down[14].

By alternating these operations (up a fifth, down a fourth to stay within an octave), Chinese theorists produced a sequence of 12 notes, identical in logic to Pythagorean tuning but expressed in pipe lengths. . The names of the 12 lü in order of generation (fifths) are recorded, for example: Huángzhōng (1), Línzhōng (a fifth up), Tàicù, Nánlǚ, Gǔxī, Yíngzhōng, Ruìbīn, Dàlǚ, Yìzé, Jiāzhōng, Wúyì, Zhōnglǚ (12th). When arranged in order of pitch within one octave, these correspond to a chromatic-like scale[1]. Early Chinese texts explicitly note the cosmological significance of completing the 12-tone cycle, associating the 12 lü with the months of the year, directions, etc., implying a full circle of fifths had mystical import[2].

Like the Pythagoreans, Chinese scholars discovered that after generating 12 fifths, one does not exactly return to a perfect octave. The 13th pitch is slightly higher than a perfect octave above the first. The Han Dynasty scholar Jing Fang (京房, 78–37 BCE) extended the cycle to 60 fifths, searching for a closer unison return[3].

He found that after 53 fifths you nearly return to the starting note (within about 0.01% frequency difference): $(3/2)^{53} \approx 2^{31}$

Error ≈ 3.6¢

In modern terms, Jing Fang recognized that 53 pure fifths ≈ 31 octaves. Jing Fang poetically called this 'the difference of one day[5].' It is astonishing that Chinese theory understood the 53-fifth near-equivalence two millennia before Europeans (Nicholas Mercator in 1660s) did[4]. This level of mathematical insight shows the sophistication of ancient Chinese acoustical science.

TWELVE LÜ AND THE PENTATONIC SCALE

Traditional Chinese music centers on the pentatonic scale—a five-note subset of the 12 lü (and of the seven-note system with added *bianyin* 變音). In Gōng mode the degrees correspond to do–re–mi–sol–la. If Huángzhōng is aligned with Western C, a common pentatonic is C–D–E–G–A (omitting F and B), obtained by the cycle of fifths—C → G → D → A → E—with octave reductions as needed. Thus, the C major pentatonic is embedded within the 12 lü generated by *sanfen sunyi* [3].

Chinese practice traditionally deemphasized the leading-tone seventh and the fourth, so the pentatonic conveniently avoids the most dissonant Pythagorean spans (major seventh, tritone). The major third (C–E) remains; in the 12-lü framework it is the Pythagorean 81:64 (≈407.8¢), brighter than a just 5:4 third. Singers may shade it slightly lower toward 5:4, and instrumental contexts can treat any beating as a familiar color rather than a defect[5]. Because Chinese music is not triad-based, a "major third" functions chiefly as a melodic step, not a chordal consonance requiring purity[4].

COMPARATIVE ANALYSIS OF TUNING SYSTEMS

By the Qing Dynasty and the European Enlightenment (18th–19th centuries), cross-cultural exchange in music theory began to recognize the connections between these tuning approaches. Both traditions confronted the Pythagorean comma and ultimately converged on equal temperament. This section compares interval sizes and consonance across just intonation, Pythagorean tuning, and 12-tone equal temperament (12-TET), using C as the reference; just and Pythagorean are shown as ratios, and cents (¢) give logarithmic sizes.

Table 3. *Intervals from C in just intonation, Pythagorean tuning, and equal temperament.* Ratios are given for just and Pythagorean; cent values (¢) indicate size on a logarithmic scale. (The "same" label indicates intervals that coincide between just and Pythagorean for those cases not involving the prime 5.)

Interval (C–*)	Just Intonation (ratio, cents)	Pythagorean (ratio, cents)	Equal Temperament (cents)
Major second (C–D)	9:8 = 1.125 (203.9¢)	9:8 = 1.125 (203.9¢) (same)	200.0¢
Major third (C–E)	5:4 = 1.25 (386.3¢)	81:64 ≈ 1.2656 (407.8¢)	400.0¢
Perfect fourth (C–F)	4:3 ≈ 1.3333 (498.0¢)	4:3 = 1.3333 (498.0¢) (same)	500.0¢
Perfect fifth (C–G)	3:2 = 1.5 (701.96¢)	3:2 = 1.5 (701.96¢) (same)	700.0¢
Major sixth (C–A)	5:3 ≈ 1.6667 (884.4¢)	27:16 = 1.6875 (905.9¢)	900.0¢
Octave (C–C')	2:1 = 2.0 (1200¢)	2:1 = 2.0 (1200¢)	1200¢

Note: In Pythagorean tuning, the major second and perfect fourth above are the same as in just intonation (since those do not involve the prime 5), whereas the intervals involving the 5th harmonic (major third, major sixth) differ significantly. Equal temperament "splits the difference" in those cases, tempering the pure just and Pythagorean intervals to a middle value[5][6].

The Chinese 12-lü corresponds to the Pythagorean column above: core melodic intervals like the fifth (3:2) and fourth (4:3) are pure. The pentatonic's major third (C–E) appears as 81:64 (≈407.8¢)—somewhat wide—yet singers often shade it lower toward 5:4 for sweetness; and because triadic harmony is not central, a wide third is typically less problematic.

Pythagorean systems can produce wolf intervals and bright thirds, spurring new temperaments in Europe. In China, heavy pentatonic use largely sidestepped these spans; with broader adoption of seven-note scales and Western influence in the 19th–20th centuries, 12-TET became standard. Notably, Zhu Zaiyu (1584) had already computed an equal-tempered division closely matching the later European adoption [7].

RESONANCE AND TIMBRE

Instrument design and practice reinforced these tunings. In Europe, the violin family (G–D–A–E) favors fifths, and flexible-pitch performers adjust intonation (e.g., narrowing major thirds in chords). In China, lutes such as pipa/ruan were historically fretted to 12-lü (many modern versions approximate 12-TET), and the guqin's harmonic markers (3/2, 4/3, 5/4) reflect overtone awareness.

Both traditions also engaged in direct experimentation. In the 11th century, the Chinese polymath Shen Kuo studied how temperature affected the pitch of lute strings and pipe lengths. In the 17th century, the French scholar Marin Mersenne precisely measured string frequencies and harmonics, laying foundations for modern acoustics. By the 19th century, European scientists such as Helmholtz explained earlier tuning practices through overtone beats and consonance theory.

CONCLUSION

European and Chinese tuning systems demonstrate a parallel development in music science. Both addressed the same mathematical problem of dividing the octave and preserving consonance, yet each prioritized intervals according to musical aesthetics. European practice gradually emphasized the ability to play in all keys, favoring even spacing over pure ratios, whereas Chinese music for centuries retained pure fifths and a pentatonic framework that reduced the need for tempering. Ultimately, both traditions adopted 12 tone equal temperament, showing that the tuning problem and its solution are universal. Musicologists view these systems not as separate worlds but as different expressions of the same idea: the interplay of mathematics, physics, and art in the pursuit of musical harmony.

REFERENCES

1. Baez, J. (2023, October 7). Pythagorean Tuning. Azimuth Blog. Retrieved from https://johncarlosbaez.wordpress.com/2023/10/07/pythagorean-tuning/
2. Crawford, Tom. (2023). The Mathematics of Music – A Closer Look into Frequencies and Tuning. Tom Rocks Maths (blog). (Discusses Pythagorean vs. equal temperament, history and math details.)
3. EarMaster Music Theory Online. (n.d.). Chapter 6.2: Tuning Systems. (Overview of various tuning systems and their basis in physics, for educational use.)
4. Gilman, B. I. (1892). On Some Psychological Aspects of the Chinese Musical System (Part I & II). The Philosophical Review, 1, 518–544, 616–627
5. Lee, Sarah. (2025). The Harmony of Just Intonation: Theoretical and Practical Aspects. Number Analytics (online article). (Explains just intonation and comparisons to other systems.)
6. Popoff, Alexandre. (2015, September 11). Tunings and temperaments. alpof (WordPress blog). Retrieved from https://alpof.wordpress.com/2015/09/11/tunings-and-temperaments/
7. Needham, J.; Robinson, K. G.; Wang, L. (1962). Science and Civilisation in China, Vol. 4, Pt. I: Physics. Cambridge University Press. See also: Cho, G. J. (2003). The Discovery of Musical Equal Temperament in China and Europe in the Sixteenth Century. Edwin Mellen Press; Peng, B. (2023). On the Western reception of Prince Zhu Zaiyu's music theory.
8. Ellis, A. J. (1885). On the Musical Scales of Various Nations. Journal of the Society of Arts, 33, 485–527; Helmholtz, H. von (trans. Ellis, A. J.) (1954). On the Sensations of Tone. Dover.
9. Helmholtz, H. von (trans. Ellis, A. J.) (1954). On the Sensations of Tone. Dover; Plomp, R., & Levelt, W. J. M. (1965). Tonal Consonance and Critical Bandwidth. JASA, 38(4), 548–560; Benade, A. H. (1990). Fundamentals of Musical Acoustics (2nd ed.). Dover.
10. Holder, W. (1694). A Treatise of the Natural Grounds and Principles of Harmony. London; Barbour, J. M. (1951/2004). Tuning and Temperament: A Historical Survey. Michigan State College Press / Dover; Muzzulini, F. (2020). Newton versus the Pythagorean Comma: a "Double Circle of Fifths." MemocS, 8(2), 101–123.
11. Boethius (trans. Bower; ed. Palisca) (1989). Fundamentals of Music. Yale; Zarlino, G. (trans. Cohen; ed. Palisca) (1983). On the Modes (Part Four of Le istitutioni harmoniche). Yale; Barbour, J. M. (1951/2004). Tuning and Temperament. Dover.
12. Zhu, Qing (translator). (2021). Triple Divisions – Key Points from the Treatises on Music (c. 680). From the Reed Ashes Pavilion (Chinese musicology blog). (Translation of an ancient Chinese text explaining sanfen sunyi.)
13. Zhu Zaiyu's Twelve Mean Law and European Music Revolution (2021 conference paper, Francis Press). (Explores the parallel invention of equal temperament by Zhu Zaiyu in China and its later adoption in Europe.)
14. Pythagorean Tuning | Azimuth https://johncarlosbaez.wordpress.com/2023/10/07/pythagorean-tuning/

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